Existence of infinitely many solutions for a class of semilinear subelliptic equations on rational Carnot groups

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Abstract
We establish the existence of infinitely many solutions for the equation
\[-\Delta_G u + u = f(\xi, u), \quad \xi \in \mathbb{G}\]
where $\Delta_G$ is a sublaplacian on a rational Carnot group $\mathbb{G}$. The function $f$ is assumed to be periodic with respect to a discrete co-compact subgroup of $\mathbb{G}$ and satisfy subcritical growth conditions.

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1 Introduction

In this paper we prove the existence of infinitely many solutions for a family of semilinear subelliptic equations, extending results obtained in the Euclidean elliptic context by Coti Zelati and Rabinowitz (see [4], [5]).

The functional setting in which we shall work is that of Carnot groups, namely simply connected nilpotent Lie groups with stratified Lie algebra (see Section 2 for notations and definitions). In particular, we are interested in Carnot groups which admit lattice subgroups.

We recall that a lattice subgroup (or simply lattice) $\Gamma$ of a nilpotent Lie group $N$ is a discrete subgroup of $N$ such that the quotient $N/\Gamma$ is compact.

According to a classical result by Mal’cev, a nilpotent Lie group contains a lattice if and only if it possesses a rational structure (see Theorem 2.2 in Section 2). Therefore, we shall confine our analysis to Carnot groups endowed with such a structure, briefly “rational” Carnot groups.
Now, let $G$ be a rational Carnot group and $\Gamma$ a lattice in $G$. The existence of a lattice enables us to give a precise notion of periodicity on $G$. We say that a function $f$ defined on $G$ is \textit{periodic under} $\Gamma$, or simply $\Gamma$-\textit{periodic}, if for any $\xi \in G$ it holds

$$f(\eta \circ \xi) = f(\xi) \quad \forall \eta \in \Gamma.$$ 

It is clear that this definition extends to the present context the usual notion of periodicity on $\mathbb{R}^n$, defined as invariance with respect to the action of the lattice subgroup $\mathbb{Z}^n$.

Let $Q$ be the homogeneous dimension of $G$ and denote by $2^* = \frac{2Q}{Q-2}$ the critical exponent for the Folland-Stein embedding on $G$. We study the following class of semilinear equations

$$-\Delta_G u + u = f(\xi, u), \quad \xi \in G$$ \hspace{1cm} (1.1)

where $\Delta_G$ denotes a fixed sublaplacian on $G$ and $f$ satisfies:

$(f_1)$ $f \in C^2(G \times \mathbb{R}, \mathbb{R})$ and $f$ is $\Gamma$-periodic in the first variable, i.e. $f(\eta \circ \xi, u) = f(\xi, u)$ for all $\eta \in \Gamma$, $\xi \in G$ and $u \in \mathbb{R}$;

$(f_2)$ $f(\xi, 0) = f_u(\xi, 0) = 0$;

$(f_3)$ There exist $a_1, a_2 > 0$ and $1 < s < \frac{Q+2}{Q-2}$ such that

$$|f_u(\xi, u)| \leq a_1 + a_2|u|^{s-1}$$

for all $\xi \in G$ and $u \in \mathbb{R}$;

$(f_4)$ There exists $\mu > 2$ such that $0 < \mu F(\xi, u) := \mu \int_0^u f(\xi, t)dt \leq f(\xi, u)u$ for all $\xi \in G$ and $u \neq 0$.

A simple example of such a function $f$ is

$$f(\xi, u) = a(\xi)|u|^{s-2}u,$$

where $a \in C^2(G, \mathbb{R})$ is $\Gamma$-periodic and $2 < s < 2^*$.

The functional corresponding to the semilinear subelliptic equation (1.1) is

$$J(u) = \frac{1}{2}\|u\|_{S^{2}_{1}}^{2} - \int_{\mathbb{R}^N} F(\xi, u) \, d\xi$$

where the norm $\|u\|_{S^{2}_{1}}$ is given by

$$\|u\|_{S^{2}_{1}}^{2} = \int_{\mathbb{R}^N} (|\nabla_G u|^2 + |u|^2) \, d\xi$$

and one seeks solutions of equation (1.1) in the Folland-Stein space $E \equiv S^{2}_{1}(G)$, defined as the completion of $C_{c}^{\infty}$ with respect to the above norm. From now on, we will simply denote by $\| \cdot \|$ the $S^{2}_{1}$-norm.
Observe that the functional \( J \) has the following property: if we let, for every \( \eta \in \Gamma \),

\[
(\tau_\eta u)(\xi) = u(\eta \circ \xi), \quad \forall \xi \in \mathbb{G},
\]

it holds

\[
J(\tau_\eta u) = J(u).
\]

As a consequence, \( J \) does not satisfy the Palais-Smale condition.

Let \( J^a = \{ u \in E | J(u) \leq a \} \), \( J_a = \{ u \in E | J(u) \geq a \} \), \( J^b = J_a \cap J^b \), where \( a, b \in \mathbb{R} \) and denote by \( K \) the set of critical points \( u \) of \( J \) in \( E \), \( K^b = J^b \cap K \), \( K_a = J^a \cap K \). Moreover, let

\[
\Lambda = \{ g \in C([0,1],E) | g(0) = 0, J(g(1)) < 0 \}
\]

and

\[
c = \inf_{g \in \Lambda} \max_{t \in [0,1]} J(g(t)).
\]

Our main result can be stated as follows.

**Theorem 1.1.** If the following condition holds:

\(*) \quad \text{there exists } \alpha > 0 \text{ such that } K^{c+\alpha}/\Gamma \text{ contains only finitely many critical points of } J, \quad \text{then } K(c) \neq \emptyset \text{ and for all } k \in \mathbb{N} \setminus \{1\}, K^{c+\alpha}_{k-\alpha}/\Gamma \text{ contains infinitely many critical points of } J. \quad \)

Observe that it is not easy to verify condition \((*)\), but of course if \((*)\) fails, then \( J \) already has infinitely many distinct critical points in \( J^{c+\alpha}/\Gamma \).

As in [5], under the assumptions of Theorem 1.1, we are able to construct infinitely many critical points of certain form, namely “multibump solutions”. More precisely, for each fixed \( k \in \mathbb{N} \) and for an appropriate choice of critical points \( v_1, \ldots, v_k \) at level \( c \), and \( \eta_1, \ldots, \eta_k \in \Gamma \) with \( d(\eta_i, \eta_j) \) sufficiently large for \( i \neq j \), it is possible to construct solutions of (1.1) near \( \sum_{i=1}^k \tau_{\eta_i} v_i \).

We conclude this introduction, pointing out that similar results in the particular case of the Heisenberg group \( \mathbb{H}^n \), the simplest non-abelian Carnot group, were proved by Maad in [12]. In that case, the function \( f \) in the right hand side of (1.1) was assumed to be invariant, in the first variable, with respect to the discrete subgroup \( \mathbb{H}^n_{\mathbb{Z}} \) consisting of the points of \( \mathbb{H}^n \) with integer coordinates.

The paper is organized as follows. Section 2 is devoted to introduce the notion of rational Carnot group and the basic notations. In Section 3 we describe the main features of the functional \( J \) under the assumptions \((f_1) - (f_4)\) and \((*)\). In particular we describe the behavior of its Palais-Smale sequences and present a deformation lemma. Then, in Section 4, we prove Theorem 1.1 as a consequence of a deeper result given by Theorem 4.4.
2 Carnot groups and lattice subgroups

In this section, we first recall the definition of Carnot group and the main properties of this functional setting. Then, we focus our attention on those Carnot groups which contain lattice subgroups.

2.1 Carnot groups

According to the classical abstract definition, a Carnot group \( \mathbb{G} \) is a connected, simply connected nilpotent Lie group of dimension \( N \geq 2 \), whose Lie algebra \( \mathfrak{g} \) admits a stratification, namely a decomposition \( \mathfrak{g} = \bigoplus_{j=1}^{r} V_j \), where \( \dim V_j = N_j \) and \( \sum_{j=1}^{r} N_j = N \), such that \([V_1, V_j] = V_{j+1}\) for \( 1 \leq j < r \), and \([V_1, V_r] = \{0\}\).

Here we shall adopt an operative definition of Carnot group, which is more commonly used in the analytic context and which is equivalent to the previous one, up to a canonical isomorphism. Consider \( \mathbb{R}^N \) as splitted in the following form \( \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \ldots \times \mathbb{R}^{N_r} \) and let \( \circ \) be an assigned Lie group law on \( \mathbb{R}^N \). Suppose \( \mathbb{R}^N \) to be endowed with a homogeneous structure by means of a family \( \{\delta_\lambda\}_{\lambda > 0} \) of group automorphisms (called dilations) of the following form

\[
\delta_\lambda(\xi) = \delta_\lambda(\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(r)}) = (\lambda \xi^{(1)}, \lambda^2 \xi^{(2)}, \ldots, \lambda^r \xi^{(r)}),
\]

where \( \xi^{(j)} \in \mathbb{R}^{N_j} \) for \( j = 1, \ldots, r \). For \( i = 1, \ldots, N_1 \), let \( X_i \in \mathfrak{g} \) be the left invariant vector field which agrees at the origin with the partial derivative \( \partial / \partial \xi_i^{(1)} \).

We require that the Lie algebra generated by \( X_1, \ldots, X_{N_1} \) is the whole \( \mathfrak{g} \). Under these assumptions, we call \( \mathbb{G} = (\mathbb{R}^N, \circ) \) a Carnot group.

The natural number \( Q = \sum_{j=1}^{r} j N_j \), naturally attached to the dilations \( \{\delta_\lambda\}_{\lambda > 0} \) is called the homogeneous dimension of \( \mathbb{G} \). The Lebesgue measure is invariant w.r.t. left and right translations on \( \mathbb{G} \) and the volumes scale as \( \lambda^Q \), i.e. \( |\delta_\lambda(E)| = \lambda^Q |E| \) for any measurable set \( E \subset \mathbb{R}^N \). In the sequel, we shall assume \( Q \geq 3 \).

Now, if \( \{Y_1, \ldots, Y_{N_1}\} \) is any basis of \( \text{span}\{X_1, \ldots, X_{N_1}\} \), the second order differential operator

\[
\Delta_{\mathbb{G}} = \sum_{i=1}^{N_1} Y_i^2
\]

is called a sublaplacian on \( \mathbb{G} \). We shall denote by \( \nabla_{\mathbb{G}} = (Y_1, \ldots, Y_{N_1}) \) the related subelliptic gradient. \( \nabla_{\mathbb{G}} \) and \( \Delta_{\mathbb{G}} \) are left-translation invariant w.r.t. the group action and \( \delta_\lambda \)-homogeneous, respectively of degree one and two. In other words, \( \nabla_{\mathbb{G}}(u \circ \tau_\xi) = \nabla_{\mathbb{G}} u \circ \tau_\xi \), \( \Delta_{\mathbb{G}}(u \circ \delta_\lambda) = \lambda \nabla_{\mathbb{G}} u \circ \delta_\lambda \), \( \Delta_{\mathbb{G}}(u \circ \tau_\xi) = \Delta_{\mathbb{G}} u \circ \tau_\xi \), \( \Delta_{\mathbb{G}}(u \circ \delta_\lambda) = \lambda^2 \Delta_{\mathbb{G}} u \circ \delta_\lambda \). Moreover, since \( X_1, \ldots, X_{N_1} \) generate the whole \( \mathfrak{g} \), any sublaplacian on \( \mathbb{G} \) satisfies Hormander’s hypoellipticity condition.

The simplest example of Carnot group is the additive group \( \mathbb{G} = (\mathbb{R}^N, +) \). In this case \( Q = N \) and the sublaplacians are exactly the constant coefficients elliptic operators on \( \mathbb{R}^N \). Moreover, if \( \mathbb{G} \) is a Carnot group of homogeneous dimension \( Q \leq 3 \), then necessarily \( \mathbb{G} \) is the ordinary Euclidean space. The simplest non-abelian Carnot group is the Heisenberg group \( \mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ) \) with homogeneous dimension...
Q = 2n+2 and composition law given by ξ ◦ ξ′ = (x+x′, y+y′, t+t′+2((x′, y)−(x, y′))) for every ξ = (x, y, t), ξ′ = (x′, y′, t′) ∈ R^{2n+1}, where x, y ∈ R^n and t ∈ R.

By definition, a homogeneous norm on G is a continuous function d : R^N → [0, +∞), smooth away from the origin, such that d(λξ) = λ d(ξ), d(ξ⁻¹) = d(ξ) and d(ξ) = 0 iff ξ = 0. Moreover, any homogeneous norm satisfies the following pseudo-triangular inequality d(ξ ◦ η) ≤ β(d(ξ) + d(η)) for a suitable constant β > 0.

If G is a Carnot group with Q ≥ 3 and Δ_G is a fixed sublaplacian on G, there exists a suitable homogeneous norm d on G defined by letting d = Γ^{1/(2-Q)}, Γ being the fundamental solution of −Δ_G with pole at 0. In the sequel, we shall call gauge distance the pseudo-distance induced by d, defined as

\[ d(ξ, η) := d(η^{-1} ◦ ξ), \quad ∀ η, ξ ∈ G, \]

and we shall use the notation B_R(ξ) for the d-ball with center at ξ and radius R.

Let us introduce the functional spaces in which we shall work. For k ∈ N and 1 ≤ p < ∞, define

\[ S_k^p(Ω) = \{ f ∈ L^p(Ω) \mid X_I f ∈ L^p(Ω) \text{ for } 0 ≤ |I| ≤ k \} \]

where I = (i_1, ..., i_k), 0 ≤ i_j ≤ N_1 for 1 ≤ j ≤ k, |I| = ∑_{i_j ≠ 0} 1 if i_j appears in I} and X_I denotes the differential operator of order |I| defined as X_I = X_{i_1}X_{i_2}...X_{i_k} (where X_0f = f). Moreover, let us recall the definition of the Hölder spaces in this context. For 0 < β < 1, let

\[ Γ^β(Ω) = \{ f ∈ C(Ω) \mid \sup_{ξ, η ∈ Ω, ξ ≠ η} \frac{|f(ξ) − f(η)|}{d(ξ, η)^β} < ∞ \}. \]

For β' = k + β, with k ∈ N and 0 < β < 1, let

\[ Γ^{β'}(Ω) = \{ f ∈ Γ^β \mid X_I f ∈ Γ^β \text{ for } 0 ≤ |I| ≤ k \}. \]

Let Ω be a bounded domain of G. Then, the following Sobolev-type embeddings, known as Folland-Stein embeddings, hold.

**Theorem 2.1.** If Ω ⊂ G is a John domain and kp < Q, then \( S_k^p(Ω) \subset L^q(Ω) \) for 1 < q ≤ \( \frac{Qp}{Qkp} \) and there exists a constant C > 0 such that

\[ \|u\|_{L^q} ≤ C\|u\|_{S_k^p}. \]

If Ω ⊂ G is a uniform domain and kp > Q, then \( S_k^p(Ω) \subset Γ^β(Ω) \) where β = k − \( \frac{Q}{p} \), and there exists a constant C > 0 such that

\[ \|u\|_{L^q} ≤ C\|u\|_{Γ^β}. \]

For the definitions and the results recalled in the theorem above see [7], [9], [14] and the extensive bibliography therein. We observe that it is not easy to show examples of domains supporting the Folland-Stein embedding in a general Carnot group.
The gauge balls, for instance, do not always enjoy the John property in Carnot groups of step higher than two, as proved in [14] by an interesting counterexample. The Carnot-Carathéodory balls, instead, are always John domains (see [9]) but the complement of such a ball is not. However, we note that, by means of a procedure introduced in [2, Lemma 1.1], exploiting the John property of the C.C. balls, any bounded domain $\Omega \subset \mathbb{G}$ can be approximated from the inside and from the outside by means of John domains.

2.2 Lattices in Carnot groups

In our analysis, we are interested in a special class of Carnot groups, namely those which admit lattice subgroups and therefore provide a natural notion of periodicity.

We recall that a lattice $\Gamma$ in a nilpotent Lie group $N$ is a discrete co-compact subgroup, namely a discrete subgroup of $N$ such that the quotient $N/\Gamma$ is compact.

The problem of the existence of lattices in nilpotent Lie groups is a classical topic. We recall here a remarkable criterion, known as Mal’cev criterion, that enables us to decide when a simply connected nilpotent Lie group admits a lattice.

**Theorem 2.2.** (A. I. Mal’cev, [13]) A simply connected nilpotent Lie group $N$ contains a lattice if and only if its Lie algebra $n$ admits a rational structure, i.e. if $n$ possesses a basis with respect to which the structure constants are rational.

(Recall, here, that, if $X = \{X_1, \ldots, X_N\}$ is a basis for the Lie algebra $n$, the structure constants of $n$ with respect to the basis $X$ are the real numbers $c_{ij}^k$ such that $[X_i, X_j] = \sum_{k=1}^N c_{ij}^k X_k$, for all $i, j = 1, \ldots, N$).

Nilpotent Lie groups endowed with a rational structure are briefly called rational.

Observe that a nilpotent Lie algebra does not always have such a structure. Only for dimensions up to 6, all nilpotent Lie algebras have $\mathbb{Q}$-structures (see e.g. [15]). On the other hand, a Lie algebra can admit different rational structures. For example, the Heisenberg algebra has uncountably many different $\mathbb{Q}$-structures (see [3, Remark p. 195]).

Now, let us focus our attention on Carnot groups containing lattices, i.e. stratified nilpotent Lie groups with rational structure. We list here some examples.

The simplest example is of course $(\mathbb{R}^N, +)$ with the lattice subgroup $\mathbb{Z}^N$.

The simplest non-abelian example is the Heisenberg group $(\mathbb{H}^n, \circ)$ and a simple example of lattice in $\mathbb{H}^n$ is the discrete subgroup $\mathbb{H}^n_\mathbb{Z}$ consisting of points with integer coordinates. A more significant example of lattice in the Heisenberg group $\mathbb{H}^1$ is the set of points $\xi_{i,j,k} = (i, j, 4k + 2ij)$, where $i, j, k \in \mathbb{Z}$ (see for instance [1] where this lattice is used to discretize a periodic problem on $\mathbb{H}^1$).

More generally, the so-called Heisenberg-type groups possess a rational structure. In fact, they possess an integral structure, as proved by Crandall and Dodziuk in [6]. Moreover, every Kolmogorov-type group constructed by means of rational matrices (see [11] for the definition) has, of course, a rational structure.

We explicitly notice that there exist stratified nilpotent Lie groups which do not admit rational structures. An interesting example of a non rational two-step group
is shown in [3, Example 5.1.13]. We mention here how this algebra is constructed. Let \( m, n \) be positive integer with \((m - 1)mn > 2(m^2 + n^2)\); for instance, we could take \( m = 6, n = 4 \). Choose real number \( c_{ij}^k \), \( 1 \leq i, j \leq m \) and \( 1 \leq k \leq n \), such that 

(i) \( c_{ij}^k = -c_{ji}^k \) for \( 1 \leq i, j \leq m \);

(ii) the \( c_{ij}^k \)'s are algebraically independent if \( i < j \).

Define \( g \) as the Lie algebra spanned by \( X_1, \ldots, X_m, Y_1, \ldots, Y_n \), where \([X_i, X_j] = \sum_{k=1}^n c_{ij}^k Y_k\) and the \( Y_i \)'s are central. Supposing, by contradiction, that \( g \) admits a basis with rational structure constants, one gets that the \( c_{ij}^k \)'s can be expressed in the following form

\[
c_{ij}^k = \sum_{p=1}^m \sum_{q=1}^m \sum_{l=1}^n \alpha_{ip} \alpha_{jq} b_{lk} \tilde{d}_{pq}, \quad \alpha_{ip}, b_{lk} \in \mathbb{R}, \tilde{d}_{pq} \in \mathbb{Q}.
\]

Thus, the \( c_{ij}^k \)'s are \( \frac{1}{2}nm(m - 1) \) numbers which are algebraically independent and belong to \( \mathbb{Q}(\alpha_{ip}, b_{lk}) \), a field with transcendence degree \( \leq m^2 + n^2 < \frac{1}{2}nm(m - 1) \), that is a contradiction.

To complete this overview, following [17], we show how a lattice can be constructed in any rational Carnot group, starting from the group law. Recall that the composition law on a Carnot group \( G \) has the following form

\[
(\xi \circ \eta)(i) = \xi(i) + \eta(i) + H(i)(\xi, \eta) \quad 1 \leq i \leq r,
\]  

(2.1)

for any \( \xi, \eta \in G \), where \( H^{(1)} \equiv 0 \) and the components of the functions \( H^{(i)} \), \( 1 < i \leq r \), are mixed polynomials in \( \xi \) and \( \eta \); moreover \( H^{(i)} \) depends only on \( \xi^{(1)}, \ldots, \xi^{(i-1)}, \eta^{(1)}, \ldots, \eta^{(i-1)} \) and \( H^{(i)}(\delta_\lambda \xi, \delta_\lambda \eta) = \lambda^i H^{(i)}(\xi, \eta) \). Now, in the case of a rational group, up to a change of coordinates, the polynomial functions \( H^{(i)} \) have rational coefficients. Let \( k_1, k_2, \ldots, k_r \) be positive integers and define \( \Gamma \subset G \) by letting \( \xi = \xi^{(i)} \in \Gamma \) if and only if \( k_i \xi^{(i)} \in \mathbb{Z} \). Due to the rationality of the coefficients of the polynomials \( H^{(i)} \), it is possible to choose the integers \( k_1, k_2, \ldots, k_r \), in such a way that \( \Gamma \) is a subgroup of \( G \). Of course, \( \Gamma \) is co-compact. Observe that \( \Gamma \) is preserved by dilations with integer parameter, i.e., \( \delta_k \Gamma \subset \Gamma \) for any \( k \in \mathbb{N} \). To fix the ideas, and without loss of generality, in our treatment we shall refer to a Carnot group \( G \) endowed with a lattice \( \Gamma \) constructed as above.

We conclude with some references. For a detailed treatment of the theory of lattices in nilpotent Lie groups, we refer the reader to the classical books [3], [15] and the monograph [16]. Related papers are also [10] and [17]. Concerning the notion of periodicity, we also point out the reference [8], where a different notion of periodicity on general Carnot groups which does not require an underlying subgroup action is adopted and which however does not work for our purposes.

### 3 Qualitative properties of the functional \( J \)

This section is devoted to the description of the main properties of the functional \( J \). We study the behavior of the Palais-Smale sequences and give a variant of the
standard deformation lemma for the sub-levels $J^b$, $b < c + \alpha$. As a consequence, we prove that, under the hypotheses ($\ast$), $c$ is a critical value of $J$. Then, we characterize the level sets $kc$, with $k \in \mathbb{N} \setminus \{1\}$, as minimax levels.

We point out that the functional $J$ has a Mountain Pass geometry, but unfortunately it does not satisfy the Palais-Smale condition, due to the lattice invariance. Hence, $c$ is not automatically a critical value for $J$.

For the sake of brevity, the proofs which mainly involve critical point theory arguments and can be straightforwardly adapted to the present context will be omitted.

**Proposition 3.1.** If $f$ satisfies $(f_1) - (f_4)$, then $J \in C^1(E, \mathbb{R})$. Moreover

$$J(u) = \frac{1}{2} \|u\|^2 + o(\|u\|^2), \quad \text{as } u \to 0.$$ 

**Remark 3.2.** By Proposition 3.1, 0 is an isolated minimum of $J$ and therefore an isolated point in $K$. Hence, there exists $\nu > 0$ such that if $u \in K \setminus \{0\}$, then $\|u\| \geq \nu$.

**Proposition 3.3.** Let $\{u_m\} \subset E$ such that $J(u_m) \to b > 0$ and $J'(u_m) \to 0$. Then, there exists $\ell \in \mathbb{N}$ (depending on $b$), $v_1, \ldots, v_\ell \in K \setminus \{0\}$, a subsequence of $u_m$ and corresponding $\{\eta^i_m\} \subset \Gamma$ such that

$$\|u_m - \sum_{i=1}^{\ell} \tau_{\eta^i_m} v_i\| \to 0. \quad (3.1)$$

Moreover

$$\sum_{i=1}^{\ell} J(v_i) = b \quad (3.2)$$

and

$$d(\eta^i_m, \eta^j_m) \to \infty, \quad \text{for } i \neq j. \quad (3.3)$$

Let $F \subset E$ be a finite set and $\ell \in \mathbb{N}$. We define

$$\mathcal{I}_\ell(F) := \left\{ \sum_{i=1}^{j} \tau_{\eta_i} v_i \mid 1 \leq j \leq \ell, v_i \in F, \eta_i \in \Gamma \right\}$$

and

$$\mu(\mathcal{I}_\ell(F)) := \inf\{\|u - w\|_{S^2_2} \mid u \neq w \in \mathcal{I}_\ell(F)\}.$$ 

It is easy to prove that $\mu(\mathcal{I}_\ell(F)) > 0$.

Assuming the hypotheses ($\ast$), choose a representative in $E$ for each equivalence class in $K^{c+\alpha}/\Gamma$, and denote the resulting set by $F$. Moreover, for any $B \subset E$, let $N_r(B) = \{u \in E \mid \|u - B\| < r\}$. A variant of the standard deformation lemma for $J^b$, $b < c + \alpha$ is now given. The proof is simply an adaptation to the present context of that of Proposition 2.60 in [5], so it will be omitted.
Proposition 3.4. (Deformation Lemma) Assume that $(f_1) - (f_4)$ and $(\ast)$ hold. If $b \in (0, c + \alpha)$, then for any $\bar{\varepsilon} \in (0, \alpha]$ and $r < \frac{1}{\mu}(\mathcal{I}_\Gamma(F))$, $\bar{\varepsilon}$ being a positive integer depending only on $\alpha$, there exist $\varepsilon \in (0, \bar{\varepsilon})$ and $\zeta \in C([0, 1] \times \mathcal{E}, \mathcal{E})$ such that

(i) $\zeta(0, u) = u$ for all $u \in \mathcal{E}$;
(ii) $\zeta(s, u) = u$ if $u \notin J^{b+\varepsilon}_b$;
(iii) $J(\zeta(s, u))$ is nonincreasing in $s$;
(iv) $\zeta(1, J^{b+\varepsilon}_b \setminus N_r(K^{b+\varepsilon}_b)) \subset J^{b-\varepsilon}_b$;
(v) $\zeta(s, \tau_\eta u) = \tau_\eta \zeta(s, u)$ for all $\eta \in \Gamma$, $s \in [0, 1]$, $u \in \mathcal{E}$.

Theorem 3.5. If $(\ast)$ holds, $c$ is a critical value of $J$.

Proof. Arguing by contradiction, suppose that $c$ is a regular value. Then, condition $(\ast)$ implies that $K^{c+\varepsilon}_c = \emptyset$ for all small $\varepsilon > 0$. Choosing any such $\varepsilon$, $r < \frac{1}{\mu}(\mathcal{I}_\Gamma(F))$, and $\varepsilon$ as given in the preceding proposition, select $g \in \Lambda$ such that

$$\max_{\theta \in [0, 1]} J(g(\theta)) \leq c + \varepsilon.$$  

Then, by (iv) of Proposition 3.4,

$$\max_{\theta \in [0, 1]} J(\zeta(1, g(\theta))) \leq c - \varepsilon. \quad (3.4)$$

On the other hand, by (ii), $\zeta(1, g) \in \Lambda$, so (3.4) contradicts the definition of $c$. \qed

Now, we give a minimax characterization of the values $kc$, $k \in \mathbb{N} \setminus \{1\}$. For $k \in \mathbb{N} \setminus \{1\}$, let

$$\Lambda_k = \{G = g_1 + \ldots + g_k \mid g_i \text{ satisfies } (G_1) - (G_3), 1 \leq i \leq k\}$$

where

$$(G_1) \ g_i \in C([0, 1]^k, \mathcal{E}) \text{ for } 1 \leq i \leq k;$$

$$(G_2) \ g_i(\theta_1, \ldots, \theta_{i-1}, 0, \theta_{i+1}, \ldots, \theta_k) = 0 \text{ and } J(g_i(\theta_1, \ldots, \theta_{i-1}, 1, \theta_{i+1}, \ldots, \theta_k)) < 0 \text{ for } 1 \leq i \leq k \text{ and for all } \theta \in [0, 1]^k;$$

$$(G_3) \text{ There exist open sets } \mathcal{O}_i, 1 \leq i \leq k, \text{ with } \overline{\mathcal{O}_i} \cap \overline{\mathcal{O}_j} = \emptyset \text{ for } i \neq j, \text{ such that supp } g_i(\theta) \subset \mathcal{O}_i \text{ for all } \theta \in [0, 1]^k.$$

Observe that starting from functions in $\Lambda$ with compact support, we can easily construct functions in $\Lambda_k$. Indeed, if $g_i \in \Lambda$, $1 \leq i \leq k$, and there are open sets $\mathcal{O}_i$, with $\overline{\mathcal{O}_i} \cap \overline{\mathcal{O}_j} = \emptyset$ for $i \neq j$, such that supp $g_i(t) \subset \mathcal{O}_i$ for all $t \in [0, 1]$, then

$$G(\theta) = \sum_{i=1}^k g_i(\theta_i) \in \Lambda_k.$$

Now, introduce the following minimax values

$$b_k = \inf_{G \in \Lambda_k} \max_{\theta \in [0, 1]^k} J(G(\theta)).$$
Proposition 3.6. $b_k = kc$.

Proof. Reasoning as in Prop. 3.4 of [5], it can be proved that if $g_i$ satisfies $(G_1), (G_2)$ for $1 \leq i \leq k$, then there exists $\tilde{\theta} \in [0, 1]^k$ such that $J(g_i(\tilde{\theta})) \geq c$, $1 \leq i \leq k$. Consequently, for each $G \in \Lambda_k$

$$J(G(\tilde{\theta})) = \sum_{i=1}^k J(g_i(\tilde{\theta})) \geq kc.$$  

Hence, $b_k \geq kc$. To prove the reverse inequality, let $\varepsilon > 0$. By means of a truncation argument, we now construct a function $\hat{g} \in \Lambda$ with supp $\hat{g}(t)$ contained in a $d$-ball centered at 0, satisfying

$$J(\hat{g}(t)) \leq c + \frac{\varepsilon}{k^2} \quad \forall t \in [0, 1],$$  

so that, appropriately choosing $\eta_i \in \Gamma$, we have that

$$G(\theta) = \sum_{i=1}^k \tau_i \hat{g}(\theta_i) \in \Lambda_k$$

and

$$J(G(\theta)) \leq kc + \varepsilon,$$

which gives $b_k = kc$. In order to construct such a $\hat{g}$, choose $g \in \Lambda$ such that

$$\max_{t \in [0, 1]} J(g(t)) \leq c + \frac{\varepsilon}{2k}. \quad (3.6)$$

Then, define

$$\hat{g}(t)(\xi) = \chi_R(d(\xi))g(t)(\xi), \quad \xi \in \mathbb{G}$$

where $R > 0$ and $\chi_R \in C^\infty(\mathbb{R}^+, \mathbb{R})$ such that $\chi_R(z) \equiv 1$ if $z \leq R$, $|\chi_R(z)| \leq 1$, and $\chi_R(z) \equiv 0$ if $z > R + 2$. Let us now verify that, taking $R$ sufficiently large, we get that $\hat{g} \in \Lambda$ and (3.5) holds. Clearly $\hat{g} \in C([0, 1], E)$ and $\hat{g}(0) = 0$. Moreover, if $\hat{g}$ satisfies

$$|J(g(t)) - J(\hat{g}(t))| < \gamma \quad \forall t \in [0, 1] \quad (3.7)$$

where

$$\gamma = \min \left( \frac{\varepsilon}{2k}, |J(g(1))| \right),$$

then $J(\hat{g}(1)) < 0$ and (3.5) follows from (3.6). To verify (3.7), observe that

$$|J(g(t)) - J(\hat{g}(t))| \leq \int_{d(\xi) > R} \left( \frac{1}{2}(|\nabla g(t)(\xi)|^2 + |g(t)(\xi)|^2) - F(\xi, g(t)) \right) d\xi$$

$$+ \int_{R < d(\xi) < R + 2} \left( \frac{1}{2}(|\nabla g(\chi_R g(t))(\xi)|^2 + |\chi_R g(t)(\xi)|^2) - F(\xi, \chi_R g(t)) \right) d\xi$$

$$\leq C \max_{t \in [0, 1]} \|g(t)\|_{L^2(B_R^C)}^2 + \int_{d(\xi) > R} F(\xi, g(t)) d\xi + \int_{R < d(\xi) < R + 2} F(\xi, \chi_R g(t)) d\xi \quad (3.8)$$
where we have used that $\nabla G(\chi_R g) = g\chi_R \nabla G d + \chi_R \nabla G g$ and that the function $\nabla G d$ is bounded, since it is homogeneous of degree 0 with respect to the dilations $\delta_\lambda$.

Hence, taking into account that
\[
|F(\xi, g(t))| \leq |g(t)|^2 + A_1 |g(t)|^{2^*},
\]
we obtain
\[
|J(g(t)) - J(\hat{g}(t))| \leq C' \max_{t \in [0,1]} \|g(t)\|_{\mathcal{S}_{1}^1(B_R)}^2 + C'' \max_{t \in [0,1]} \|g(t)\|_{L^{2^*}(B_R)}^{2^*} \equiv \psi(R),
\]
where $\psi(R) \to 0$ as $R \to \infty$. So, taking $R$ sufficiently large, we get (3.7) and the proof is complete.

\section{Existence of infinitely many solutions}

In this section we prove that, under the assumption $(\ast)$, $\kappa^{k \alpha + \alpha n}/\Gamma$ is an infinite set for all $k \in \mathbb{N} \setminus \{1\}$. Actually, this will be done in such a way to give more information about the form of the solutions.

First, we introduce a finite set $A$ of critical points at level $c$, with the property that, in a sufficiently small neighborhood of $A$ we can select functions $g \in \Lambda$ which “approximate” $c$ in a suitable sense. More precisely, let
\[
\alpha_1 = \sup \{ \beta < \alpha | \kappa^{\alpha + \beta} = \mathcal{K}(c) \}. \tag{4.1}
\]
Then, the following holds.

**Proposition 4.1.** Assume that $(f_1) - (f_4)$ and $(\ast)$ hold. Then there exists a finite set $A \subset \mathcal{K}(c)$ with the property that, for any $\varepsilon_1 < \frac{\alpha_1}{2}$, $r_1$ sufficiently small and $p \in \mathbb{N}$, there exist $\varepsilon_1 \in (0, \varepsilon_1)$ and $g_1 \in \Lambda$ such that
\begin{enumerate}[(i)]
  
  \item $\max_{t \in [0,1]} J(g_1(t)) \leq c + \frac{\varepsilon_1}{p};$
  
  \item If $J(g_1(t)) > c - \varepsilon_1$, then $g_1(t) \in N_{r_1}(A)$.
\end{enumerate}

**Proof.** The proof is analogous to that of [4, Prop. 2.22].

Consider, now, $\eta_1, \ldots, \eta_k \in \Gamma$ such that for $i \neq j$,
\[
d(\eta_i, \eta_j) \geq n_0 \in \mathbb{N}, \tag{4.2}
\]
where $n_0$ is sufficiently large that
\[
\| \sum_{i=1}^{k} \tau_{\eta_i} v_i \| \geq k^{\nu} \tag{4.3}
\]
for all choices of $v_i \in A$ and $\nu$ as defined in Remark 3.2. Moreover, let
\[
\mathcal{M}(\eta_1, \ldots, \eta_k, A) := \left\{ \sum_{i=1}^{k} \tau_{\eta_i} v_i \mid v_i \in A \right\}.
\]
\[11\]
Proposition 4.2. If (*) is satisfied, there exists \( r_k = r_k(\alpha) \) and \( n_0 = n_0(A, \alpha) \) in (4.2) such that if \( r \leq r_k \) and \( w \in \overline{N}_r(\bigcup_{\ell \in \mathbb{N}} \mathcal{M}(\delta_\ell \eta_1, \ldots, \delta_\ell \eta_k, A)) \cap K \), then \( w \in K_{kc+\alpha} \).

Proof. We only note that \( n_0 \) is chosen to satisfy (4.3) and \( |J(\sum_{i=1}^k \tau_\delta \eta_i v_i) - kc| < \alpha \) for all \( v_i \in A \) and \( \ell \in \mathbb{N} \).

Proposition 4.3. Let (*) be satisfied. Then, there exists \( r_0 > 0 \) such that, for any \( r < r_0 \), the following alternative holds:

(i) \( \exists \delta_\ell > 0 \) such that \( \|J(w)\| \geq \delta_\ell \) for all \( w \in N_r(\mathcal{M}(\delta_\ell \eta_1, \ldots, \delta_\ell \eta_k, A)) \), or

(ii) there is a \( w \in \overline{N}_r(\mathcal{M}(\delta_\ell \eta_1, \ldots, \delta_\ell \eta_k, A)) \) such that \( J(w) = 0 \).

Moreover, if

\[ L = \{ \ell \in \mathbb{N} \mid (i) \text{ holds for } \mathcal{M}(\delta_\ell \eta_1, \ldots, \delta_\ell \eta_k, A) \} \]

and

\[ W = \bigcup_{\ell \in L} \mathcal{M}(\delta_\ell \eta_1, \ldots, \delta_\ell \eta_k, A) \]

then \( \exists \delta > 0 \), independent of \( \ell \), such that \( \|J(w)\| \geq \delta \) for all \( w \in N_r(W) \setminus N_{r/8}(W) \).

Theorem 4.4. Let (*) be satisfied. Then, for any sufficiently small \( r > 0 \), and for any choice of elements \( \eta_1, \ldots, \eta_k \in \Gamma \), with corresponding \( n_0 \) as in Proposition 4.2, it holds

\[ K_{kc+\alpha}/\Gamma \cap N_r(\mathcal{M}(\delta_\ell \eta_1, \ldots, \delta_\ell \eta_k, A)) \neq \emptyset \]

(4.5)

for all but finitely many \( \ell \in \mathbb{N} \).

Remark 4.5. Observe that Theorem 1.1 easily follows from the above theorem, taking into account that the sets \( N_r(\mathcal{M}(\delta_\ell \eta_1, \ldots, \delta_\ell \eta_k, A)) \) and \( N_r(\mathcal{M}(\delta_\ell \eta_1, \ldots, \delta_\ell \eta_k, A)) \) are disjoint if \( |\ell - \ell'| \) is sufficiently large. Hence, under the assumption (*), \( K_{kc+\alpha}/\Gamma \) contains infinitely many critical points for all \( k \in \mathbb{N} \setminus \{1\} \).

Proof of Theorem 4.4 To prove the theorem, it suffices to show that \( L \) as defined in (4.4) is finite. Arguing by contradiction, suppose that \( L \) is infinite. Let \( r < r_0 \) and \( \delta \) be as given in Proposition 4.3, and choose \( \varepsilon_1 > 0 \) such that

\[ \varepsilon_1 < \min \left( \frac{r \delta}{32}, \frac{\alpha_1}{2} \right) \]

with \( \alpha_1 \) as defined in (4.1). By Proposition 4.1, there exists \( \epsilon = \frac{\alpha_1}{2} \in (0, \frac{\alpha_1}{2}) \) and \( g_1 \in \Lambda \) such that

\[ \max_{t \in [0,1]} J(g_1(t)) \leq c + \frac{\epsilon}{3k} \]

and

\[ g_1(t) \in N_{infty}(A), \text{ if } J(g_1(t)) > c - 2\epsilon. \]

By using a suitable cut-off function as in Proposition 3.6, we get a function \( g \in \Lambda \) and \( R > 0 \) such that

\[ \|g_1(t) - g(t)\| \leq \frac{r}{16k}, \]
\[ |J(g_1(t)) - J(g(t))| < \frac{\varepsilon}{6k} \]

and

\[ \text{supp } g(t) \subset B_{\frac{r}{2}}(0), \quad \text{for all } t \in [0, 1]. \]

Hence

\[ \max_{t \in [0, 1]} J(g(t)) \leq c + \frac{\varepsilon}{2k}, \]

and \( J(g(t)) > c - \frac{3\varepsilon}{2} \) implies \( g(t) \in N_{\frac{r}{8\sqrt{k}}}(A) \).

Let \( \beta > 0 \) be a number which is free for the moment and will be specified later in terms of \( r \). Let \( \eta_1, \ldots, \eta_k \in \Gamma \) with \( n_0 \) as given by Proposition 4.2, and choose \( \tilde{\ell} \in \mathcal{L} \) so that

\[ d \left( \tau_{\delta_\eta_i}(B_R(0)), \tau_{\delta_\eta_j}(B_R(0)) \right) \geq \beta + 2 \quad \text{for } i \neq j. \quad (4.6) \]

Observe that, since \( \mathcal{L} \) is infinite, for any given \( \beta > 0 \), we can choose \( \tilde{\ell} \in \mathcal{L} \) satisfying (4.6). Now, denote \( \tilde{\eta}_i = \delta_{\tilde{\ell}} \eta_i \) and define, for \( \theta \in [0, 1]^k \),

\[ G(\theta) = \sum_{i=1}^{k} \tau_{\tilde{\eta}_i} g(\theta_i). \]

Hence

\[ \text{supp } G(\theta) \subset \bigcup_{i=1}^{k} B_{\frac{r}{2}}(\tilde{\eta}_i) \]

and

\[ J(G(\theta)) = \sum_{i=1}^{k} J(g(\theta_i)) \leq kc + \varepsilon, \quad \text{for all } \theta \in [0, 1]^k. \]

Moreover, if \( J(G(\theta)) > kc - \varepsilon \), then \( G(\theta) \in N_{\frac{r}{8\sqrt{k}}}(M(\tilde{\eta}_1, \ldots, \tilde{\eta}_k, A)) \). In particular, since \( \tilde{\ell} \in \mathcal{L} \), \( J(G(\theta)) > kc - \varepsilon \) implies \( G(\theta) \in N_{\frac{r}{8\sqrt{k}}}(W) \).

Throughout the remainder of the proof, the function \( G \) will be subsequently modified in order to get a function \( H \in \Lambda_k \) satisfying

\[ \max_{\theta \in [0, 1]^k} J(H(\theta)) \leq kc - \varepsilon, \]

that is a contradiction, due to Proposition 3.6.

First, by means of a flow corresponding to an appropriate pseudogradient vector field, following the Euclidean outline (see Step 2 in [5, Section 4]) \( G \) is homotoped to a function \( \tilde{G} \) such that

\[ \max_{\theta \in [0, 1]^k} J(\tilde{G}(\theta)) \leq kc - \varepsilon. \]

If \( \tilde{G} \in \Lambda_k \), we would have a contradiction at this point. Unfortunately, we only know that

\[ \| \tilde{G} \|_{S^1_{\mathcal{L}}(\mathbb{R}^N \setminus \bigcup_{i=1}^{k} B_{\frac{r}{2}}(\tilde{\eta}_i))} \leq r. \]
By using a truncation argument similar to that used in Proposition 3.6, starting from \( \hat{G} \) we obtain a smooth function \( \hat{G} \) with compact support contained in a \( d \)-ball \( B_{\hat{R} + 2}(0) \), such that
\[
\| \hat{G}(\theta) - G(\theta) \| \leq 2r \tag{4.7}
\]
and
\[
\max_{\theta \in [0,1]^k} J(\hat{G}(\theta)) \leq kc - \varepsilon/4. \tag{4.8}
\]

Let \( V_i = B_{\hat{R}}(\tilde{\eta}_i) \). We can always assume \( \hat{R} > 0 \) so large that \( V_i \subset B_{\hat{R} + 2}(0) \) and \( d(\partial B_{\hat{R} + 2}(0), V_i) \geq \min_{j \neq i} d(V_i, V_j) \) for \( 1 \leq i \leq k \). We now construct a function \( v \) which shall replace \( \hat{G} \) outside of the sets \( V_i \). Let
\[
V = B_{\hat{R} + 2}(0) \setminus \bigcup_{i=1}^k V_i.
\]

Observe that \( V \) could not be a John domain, since the gauge balls and their complements do not always enjoy the John property in Carnot groups of step higher than two (see [14]). However, by using the argument introduced in [2, Lemma 1.1], for any \( \delta > 0 \), it is always possible to construct a John domain \( V_\delta \subset V \) such that \( \{ \xi \in V | d(\xi, \partial V) > \delta \} \subset V_\delta \subset V \). Let, then, \( W \) be a John domain such that
\[
\left\{ \xi \in V | d(\xi, \partial V) > \frac{1}{2} \right\} \subset W \subset V,
\]
and introduce the set
\[
\hat{E}(\theta) = \{ v \in S_1^2(W) | v - \hat{G}(\theta) \in S_1^2(W), \| v \|_{S_1^2(W)} < 8r \}.
\]

Note that \( \hat{E}(\theta) \neq \emptyset \). Indeed, since by (4.7),
\[
\| \hat{G}(\theta) \|_{S_1^2(W)} \leq \| \hat{G}(\theta) \|_{S_1^2(V)} = \| \hat{G}(\theta) - G(\theta) \|_{S_1^2(V)} \leq 2r,
\]
then \( \hat{G}(\theta) \mid W \in \hat{E}(\theta) \).

Consider, now, the functional
\[
\Psi(v) = \int_W \left( \frac{1}{2} (|\nabla_\xi v|^2 + v^2) - F(\xi, v) \right) d\xi.
\]

**Lemma 4.6.** There exists a unique function \( v = v(\theta) \in \hat{E}(\theta) \) which minimizes \( \psi \). Moreover, \( v \) continuously depends on \( \theta \) in \( \| \cdot \|_{S_1^2(W)} \).

**Proof.** It can be easily seen that, if the infimum of \( \Psi \) over \( \hat{E}(\theta) \) is attained at some \( v \), then \( \| v \|_{S_1^2(W)} \leq 4r \), i.e. \( v \) is an interior point of \( \hat{E}(\theta) \). From this property and the weak lower semicontinuity of the functional \( \Psi \), the existence of a minimum \( v \) follows. Observe that the function \( v \) is a weak solution of
\[
\begin{cases}
-\Delta_G v + v = f(\xi, v), & \xi \in W \\
v = \hat{G}(\theta), & \xi \in \partial W.
\end{cases} \tag{4.9}
\]
To show uniqueness, let \( v \) and \( w \) be solutions of (4.9). Then

\[
\|v-w\|_{S_2^2(W)}^2 = \int_W (f(\xi, v) - f(\xi, w))(v - w) \, d\xi \\
= \int_W (v - w)^2 \left( \int_0^1 f_u(\xi, w + t(v - w)) \, dt \right) \, d\xi.
\]

(4.10)

By the growth conditions on \( f \), there exists a constant \( A_8 > 0 \) such that

\[
|f_u(\xi, z)| \leq \frac{1}{8} + A_8 |z|^{2r-2}.
\]

Hence, by Hölder’s inequality and by the Folland-Stein embedding on \( W \), we get

\[
\|v-w\|_{S_2^2(W)}^2 \leq \frac{1}{8} \|v-w\|_{S_2^1(W)}^2 + A_8 \int_W (v - w)^2 \left( \|v\| + |w| \right)^{2r-2} \, d\xi \\
\leq \frac{1}{8} \|v-w\|_{S_2^2(W)}^2 + A_8 C^2 \|v-w\|_{S_2^2(W)}^2 \left( \|v\|_{L^2(W)}^2 + \|w\|_{L^2(W)}^2 \right)^{2r-2} + C \|w\|_{S_2^1(W)}^2 \]

\[
\leq \frac{1}{8} \|v-w\|_{S_2^2(W)}^2 + A_8 C^2 (8r)^{2r-2} \|v-w\|_{S_2^1(W)}^2,
\]

(4.11)

from which, requiring that \( r \) satisfies

\[
A_8 C^2 (8r)^{2r-2} < \frac{7}{8},
\]

we get that \( v = w \).

Finally, it follows from uniqueness that \( v \) continuously depends on \( \theta \). \qed

Now we prove an interior \( L^\infty \)-estimate for \( v \).

**Lemma 4.7.** Let \( D_\rho = \{ \xi \in W \mid d(\xi, \partial W) \geq \rho \} \). Then, there exists a constant \( K > 0 \) depending only on \( \rho \) and \( Q \), such that

\[
\|v\|_{L^\infty(D_\rho)} \leq K \|v\|_{S_2^2(W)}.
\]

(4.12)

**Proof.** We shall use a bootstrap argument. Let \( O \subset \subset \tilde{O} \subset \subset W \). Then, there exists a constant \( K_1 > 0 \) such that the following local \( L^p \)-estimate holds for \( v \):

\[
\|v\|_{S_2^2(O)} \leq K_1 (\|f(\cdot, v)\|_{L^p(\tilde{O})} + \|v\|_{L^p(\tilde{O})}),
\]

(4.13)

where \( K_1 \) depends on \( p, Q, \mathrm{diam} \tilde{O} \) and \( d(O, \tilde{O}) \) (see [7]). Let \( \xi \in D_\rho \) and \( B_i = B_{i\rho/2j}(\xi), i = 1, \ldots, j + 1 \), with \( j \in \mathbb{N} \) free for the moment. We shall use estimate (4.13) with \( O = B_i \) and \( \tilde{O} = B_m, m > i \).
Let \( p_0 = 2^*s^{-1} \), where recall that \( 1 < s < 2^* - 1 \). Then
\[
\| f(\cdot, v) \|_{L^{p_0}(B_{j+1})} \leq \| a_1|v| + a_2s^{-1}|v|^s \|_{L^{p_0}(B_{j+1})}
\leq a_1\|v\|_{L^{p_0}(B_{j+1})} + a_2s^{-1}\|v\|^s_{L^{2^*}(B_{j+1})} \leq C\|v\|_{S_1^2(W)},
\]
where we have used that \( \|v\|_{S_1^2(W)} \leq 8r < 1 \). Hence, by (4.13) we get
\[
\|v\|_{S_2^{p_0}(B_j)} \leq C\|v\|_{S_1^2(W)}.
\]
If \( p_0 > Q \), taking \( j = 2 \) and by applying the Folland-Stein embedding, from (4.15) we obtain
\[
\|v\|_{L^\infty(B_1, t)} = \|v\|_{L^\infty(B_{p/4}(\xi))} \leq C\|v\|_{S_1^2(W)}
\]
and we have done, due to the arbitrariness of \( \xi \in D_{p/4} \).

If \( p_0 = \frac{Q}{2} \), take \( j = 3 \) and let \( \bar{p} < p_0 \) be given by
\[
\frac{1}{s(\frac{Q}{2} + 1)} = \frac{1}{\bar{p}} - \frac{2}{Q}.
\]
Thus, by the Folland-Stein embedding and taking into account (4.15), we get
\[
\|v\|_{L^{s(Q/2+1)}(B_j)} \leq C\|v\|_{S_1^2(B_{j+1})} \leq C\|v\|_{S_1^2(W)}
\]
and consequently, as above, we have
\[
\|f(\cdot, v)\|_{L^{Q/2+1}(B_j)} \leq C\|v\|_{S_1^2(W)}.
\]
Therefore, by (4.13), it follows
\[
\|v\|_{S_2^{Q/2+1}(B_{j-1})} \leq C\|v\|_{S_1^2(W)},
\]
from which
\[
\|v\|_{L^\infty(B_1, t)} = \|v\|_{L^\infty(B_{p/6}(\xi))} \leq C\|v\|_{S_1^2(W)}.
\]
If \( p_0 < \frac{Q}{2} \), let \( t_0 > 0 \) be such that
\[
\frac{1}{t_0} = \frac{1}{p_0} - \frac{2}{Q}.
\]
It is easy to check that \( t_0 > 2^* \). We choose \( j \) so that
\[
js\left(\frac{1}{2^*} - \frac{1}{t_0}\right) > \frac{1}{t_0}.
\]
Now,
\[
\|v\|_{L^{p_0}(B_j)} \leq C\|v\|_{S_2^{p_0}(B_{j+1})} \leq C\|v\|_{S_1^2(W)},
\]
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so that, letting $p_1 = t_0 s^{-1} > 2^* s^{-1} = p_0$, we get
\[ \| f(\cdot, v) \|_{L^p(B_j)} \leq C \| v \|_{S_2^1(W)}. \] (4.18)
Then, by (4.13),
\[ \| v \|_{S_2^1(B_{j-1})} \leq C \| v \|_{S_2^1(W)}. \] (4.19)
Arguing as earlier, if $p_1 > \frac{Q}{2}$ or $p_1 = \frac{Q}{2}$, from (4.19) we get
\[ \| v \|_{L^\infty(B_1)} \leq \| v \|_{L^\infty(B_{j-2})} \leq C \| v \|_{S_2^1(W)}. \] (4.20)
If $p_1 < \frac{Q}{2}$, we continue this process with
\[ \frac{1}{t_i} = \frac{1}{p_i} - \frac{2}{Q}, \quad p_{i+1} = t_i s^{-1}. \]
It turns out that in at most $j$ steps, we find $p_j \geq \frac{Q}{2}$ and, reasoning as above,
\[ \| v \|_{L^\infty(B_1)} = \| v \|_{L^\infty(B_{j/2}(\xi))} \leq C \| v \|_{S_2^1(W)}. \] (4.21)
Indeed, if $p_j < \frac{Q}{2}$, then
\[ 0 < \frac{1}{t_j} = \frac{1}{p_j} - \frac{2}{Q} = \sum_{i=1}^{j} \left( \frac{1}{p_i} - \frac{1}{p_{i-1}} \right) + \frac{1}{p_0} - \frac{2}{Q} \leq js \left( \frac{1}{t_0} - \frac{1}{2^*} \right) + \frac{1}{t_0}, \] contrary to (4.17). The proof is therefore complete. \(\square\)

Now, we prove that $v$ can be made exponentially small in certain annular regions surrounding each set $V_i = B_R(\tilde{\eta}_i)$. Introduce the sets
\[ \hat{S}_i = \{ \xi \in G \mid R + 1 \leq d(\tilde{\eta}_i, \xi) \leq R + \beta + 1 \}, \quad 1 \leq i \leq k \]
and their subsets
\[ A_i = \{ \xi \in \hat{S}_i \mid R + \frac{\beta}{2} - 1 < d(\tilde{\eta}_i, \xi) < R + \frac{\beta}{2} + 1 \}, \quad 1 \leq i \leq k. \] (4.22)
We prove the following estimate.

**Lemma 4.8.** Let $v(\theta)$ be the minimizer obtained in Lemma 4.6. Then, there exists $\gamma > 0$ such that the following estimate holds
\[ |v(\theta)(\xi)| \leq C e^{-\gamma \frac{\beta}{2}}, \quad \forall \xi \in A_i. \] (4.23)
Proof. First observe that by (f3), there exists \( r > 0 \) such that, if \(|u| \leq r\), then \(|f(\xi, u)| \leq \frac{1}{2}|u|\). Choosing \( \rho = 1/2 \) in (4.12) and \( r \) so that

\[
r \leq \frac{r}{8K},
\]

we get

\[
\|v\|_{L^\infty(D_{1/2})} \leq r.
\] (4.24)

Note that \( \hat{S}_i \subset D_{1/2} \). Taking into account that

\[
-\Delta_G v^2 + v^2 = -2|\nabla_G v|^2 - 2v\Delta_G + v^2
\]

and since, by (4.24), \(|f(\xi, v)| \leq \frac{1}{2}|v|\) in \( D_{1/2} \), we get

\[
\left\{ \begin{array}{ll}
-\Delta_G v^2 + v^2 & \leq 0, \quad \xi \in \hat{S}_i, \\
v^2 & \leq r^2, \quad \xi \in \partial \hat{S}_i.
\end{array} \right.
\] (4.25)

In order to estimate \( v \), we now construct a suitable function \( w \) satisfying

\[
\left\{ \begin{array}{ll}
-\Delta_G w + w & \geq 0, \quad \xi \in \hat{S}_i \\
w & \geq v^2, \quad \xi \in \partial \hat{S}_i.
\end{array} \right.
\] (4.26)

Hence

\[
\left\{ \begin{array}{ll}
-\Delta_G (w - v^2) + (w - v^2) & \geq 0, \quad \xi \in \hat{S}_i \\
w - v^2 & \geq 0, \quad \xi \in \partial \hat{S}_i,
\end{array} \right.
\] (4.27)

so that, by the weak maximum principle applied to \(-\Delta_G + I\), we get

\[
w \geq v^2, \quad \xi \in \hat{S}_i.
\] (4.28)

For simplicity, assume that \( \tilde{h}_i = 0 \). Let us define

\[
w_{\pm} = e^{\pm \gamma d(\xi, 0)}, \quad \gamma > 0.
\]

A calculation shows that, for \( \gamma > 0 \) sufficiently small, then

\[
-\Delta_G w_{\pm} + w_{\pm} \geq 0, \quad \xi \in \hat{S}_i.
\] (4.29)

Indeed, if we compute \( \Delta_G w_{\pm} \) by means of the formula: \( \Delta_G f = \psi \left( f'' + \frac{Q-1}{d} f' \right) \), \( \psi := |\nabla_G d|^2 \), which holds for any function \( f = f(d) \) on \( G \) only depending on the homogeneous norm \( d \), we easily get

\[
\Delta_G w_{\pm} = \psi \left( \gamma^2 \pm \frac{Q-1}{d} \gamma \right) w_{\pm}.
\]
Hence, remembering that \(\psi\) is a bounded function and that \(d(\xi) \geq R + 1\) for \(\xi \in \hat{S}_i\), we get
\[-\Delta_G w_\pm + w_\pm = \left[-\psi \left(\gamma^2 \pm \frac{Q-1}{d(\xi)} \right) + 1\right] w_\pm \geq 0, \quad \xi \in \hat{S}_i,
\]
choosing \(\gamma > 0\) sufficiently small.

Now, let
\[
w = r_2^2 \left(e^{-\gamma(R+\beta+1)} w_+ + e^{\gamma(R+1)} w_-\right) = r_2^2 \left(e^{-\gamma((R+\beta+1)-d(\xi,0))} + e^{\gamma((R+1)-d(\xi,0))}\right).
\]
(4.30)

By (4.29) and taking into account that \(w = r_2^2(e^{-\gamma\beta} + 1)\) on \(\partial \hat{S}_i\), it follows that \(w\) satisfies (4.26). Now, by (4.30) we have that
\[
w|_{d(\xi)=R+\frac{\beta}{2}+1} = 2r_2^2 e^{-\gamma\frac{\beta}{2}} \cosh \gamma t,
\]
from which, taking into account (4.28),
\[v^2(\xi) \leq 2r_2^2 e^{-\gamma\frac{\beta}{2}} \cosh 2\gamma, \quad \forall \xi \in A_i,
\]
that is the desired estimate. \(\square\)

For each \(\theta \in [0, 1]^k\), define
\[
U(\theta)(\xi) = \begin{cases} 
\hat{G}(\theta)(\xi) & \xi \notin W \\
v(\theta)(\xi) & \xi \in W. 
\end{cases}
\]
By (4.8) and by the definition of \(v\), we get
\[
J(U(\theta)) \leq J(\hat{G}(\theta)) \leq kc - \frac{\varepsilon}{4}. \tag{4.31}
\]
Now, we shall modify \(U\) in the annular regions \(A_i\) defined in (4.22) in order to construct a function \(H \in \Lambda_k\) such that
\[
\max_{\theta \in [0,1]^k} J(H(\theta)) \leq kc - \frac{\varepsilon}{8}. \tag{4.32}
\]
For \(1 \leq i \leq k\), let
\[
h_i(\theta)(\xi) = \begin{cases} 
U(\theta)(\xi), & \text{if } d(\tilde{\eta}_i, \xi) < R + \frac{\beta}{2} \\
2 \left|d(\tilde{\eta}_i, \xi) - \left(R + \frac{\beta}{2} + 1\right)\right| U(\theta)(\xi), & \text{if } R + \frac{\beta}{2} < d(\tilde{\eta}_i, \xi) < R + \frac{\beta}{2} + 1 \\
0 & \text{otherwise.}
\end{cases}
\]
Then, define
\[
H(\theta) = \sum_{i=1}^k h_i(\theta).
\]
If we require, e.g., $\beta > 1$, it is easily seen that $H \in \Lambda_k$. Now, to prove (4.32), taking into account (4.31), it is sufficient to show that

$$|J(H(\theta)) - J(U(\theta))| \leq \frac{\varepsilon}{8}. \quad (4.33)$$

Note that $H \equiv U$ on the set $W = \bigcup_{i=1}^k B_{R+\frac{\beta}{2}}(\tilde{\eta}_i)$. Hence, letting $D = W \setminus W$, (4.33) follows if we prove that

$$\left| \int_D \left( \frac{1}{2}(|\nabla_G H|^2 + H^2) - F(\xi, H) \right) d\xi \right| + \left| \int_D \left( \frac{1}{2}(|\nabla_G U|^2 + U^2) - F(\xi, U) \right) d\xi \right| \equiv J_H + J_U \leq \frac{\varepsilon}{8}.$$

Let us begin by estimating the term $J_H$. Let

$$C_i = B_{R+\frac{\beta}{2} + \frac{1}{2}}(\tilde{\eta}_i) \setminus B_{R+\frac{\beta}{2}}(\tilde{\eta}_i).$$

Then,

$$J_H \leq \sum_{i=1}^k \left| \int_{C_i} \frac{1}{2}(|\nabla_G h_i|^2 + h_i^2) - F(\xi, h_i) d\xi \right| \leq \frac{1}{2} \sum_{i=1}^k \|h_i\|_{S^2_{p}(C_i)}^2, \quad (4.34)$$

where we have used that, for $\beta$ sufficiently large, we get $|v(\theta)(\xi)| \leq 1$ in $C_i$, and subsequently $F(\xi, h_i) \leq \frac{1}{2}h_i^2$ for any $\xi \in C_i$.

Let, now, $\xi \in C_i$. Taking $O = B_{\frac{1}{4}}(\xi)$ and $\tilde{O} = B_{\frac{1}{2}}(\xi)$ in (4.13), and using (4.23), we have

$$\|v\|_{S^p_{\frac{1}{4}}(O)} \leq Ke^{-\gamma \frac{\beta}{4}},$$

which implies, choosing $p > Q$,

$$\|v\|_{S^p_{\frac{1}{4}}(O)} \leq K' e^{-\gamma \frac{\beta}{4}}, \quad (4.35)$$

where $\theta = 2 - \frac{Q}{p} > 1$. Hence, from (4.34) and (4.35), which holds for arbitrary $\xi \in C_i$, we get

$$J_H \leq Ce^{-\gamma \frac{\beta}{4}} \sum_{i=1}^k \text{vol} C_i \leq C \left( R + \frac{\beta}{2} + \frac{1}{2} \right)^Q e^{-\gamma \frac{\beta}{4}},$$

so that, choosing $\beta$ sufficiently large, we obtain

$$J_H \leq \frac{\varepsilon}{16}.$$
Let us now estimate $J_U$. Since, by $(f_3) - (f_4)$, there exists $A_4 > 0$ such that

$$F(\eta, u) \leq \frac{1}{4}|u|^2 + A_4|u|^{2^*},$$

we get

$$\int_{\mathcal{D}} F(\xi, v) \, d\xi \leq \frac{1}{4} \int_{\mathcal{D}} v^2 \, d\xi + A_4 \int_{\mathcal{D}} |v|^{2^*} \, d\xi \leq \left( \frac{1}{4} + A_4 \|v\|_{\mathcal{S}_1^2(W)}^{2^* - 2} \right) \|v\|_{\mathcal{S}_1^2(\mathcal{D})}^2. \quad (4.36)$$

Recalling that $\|v\|_{\mathcal{S}_1^2(W)} \leq 4r$, and requiring that $r$ satisfies

$$\frac{1}{4} + A_4 (4r)^{2^* - 2} \leq \frac{1}{2},$$

from (4.36) we get

$$J_U \leq \frac{1}{2} \|v\|_{\mathcal{S}_1^2(\mathcal{D})}^2. \quad (4.37)$$

Then, multiplying (4.9) by $v$ and integrating over $\mathcal{D}$, we have

$$\|v\|_{\mathcal{S}_1^2(\mathcal{D})}^2 = \int_{\mathcal{D}} vf(\xi, v) \, d\xi - \int_{\partial \mathcal{D}} v < A \nabla v, \tilde{n} > \, d\Theta, \quad (4.38)$$

where $\tilde{n}$ is the outward unit normal to $\partial \mathcal{D}$ at $\xi$ and $A$ is the positive semidefinite matrix such that $\Delta_G u = \text{div}(A \nabla u)$. By the growth conditions on $f$, it follows that

$$\int_{\mathcal{D}} vf(\xi, v) \, d\xi \leq \frac{1}{2} \|v\|_{\mathcal{S}_1^2(\mathcal{D})}^2.$$

Moreover, since $\partial \mathcal{D} = \partial B_{R+2}(0) \cup \left( \bigcup_{i=1}^{k} \partial B_{R+\frac{\beta}{2}}(\tilde{\eta}_i) \right)$, observing that $v \equiv 0$ on $\partial B_{R+2}(0)$ and that, by (4.35),

$$|v(\xi)|, |\nabla_G v(\xi)| \leq Ke^{-\gamma \frac{\beta}{2}}, \quad \text{for } \xi \in \bigcup_{i=1}^{k} \partial B_{R+\frac{\beta}{2}}(\tilde{\eta}_i),$$

from (4.38) we get

$$\frac{1}{2} \|v\|_{\mathcal{S}_1^2(\mathcal{D})}^2 \leq K(R + \frac{\beta}{2})^{Q-1} e^{-\gamma \frac{\beta}{2}}.$$

This estimate, combined with (4.37), implies, for $\beta$ sufficiently large,

$$J_U \leq \frac{\epsilon}{16},$$

and this completes the proof. \qed
Remark 4.9. We observe that our results also apply if the operator $-\Delta_G + I$ in (1.1) is replaced by a more general divergence structure operator on $G$ of the form

$$Lu = -\sum_{i,j=1}^{N_1} X_i(a_{ij}(\xi) X_j u) + b(\xi) u,$$

where $X = (X_1, \ldots, X_{N_1})$ denotes any basis of the first layer of the Lie algebra of $G$ and the coefficients $a_{ij}$ and $b$ satisfy the following conditions:

(i) $a_{ij} = a_{ji} \in C^2(\mathbb{R}^N)$, $b \in C^1(\mathbb{R}^N)$ and $a_{ij}, b$ are $\Gamma$-periodic;
(ii) there exists a constant $C > 0$ such that

$$\sum a_{ij}(\xi) \eta_i \eta_j \geq C|\eta|^2, \quad \xi, \eta \in \mathbb{R}^N;$$

(iii) $b(\xi) > 0$, $\forall \xi \in \mathbb{R}^N$.

In fact, the proof is essentially the same with only a few technical modifications. The Euclidean outline can be found in [5, Section 6].

References


